HERMITE-HADAMARD TYPE INEQUALITIES FOR MULTIDIMENSIONAL PREINVEX STOCHASTIC PROCESSES

Nurgül Okur[1] and Vildan Karahan[2]
[1] Giresun University, Department of Statistics, nrgokur@gmail.com
[2] Institute of Science, Giresun University, vildan.karahan2@ugurokullari.k12.tr

ABSTACT

In this paper, we presented initially preinvex stochastic processes on real number set, and obtained Hermite-Hadamard type inequality for these processes. Consequently, we identified multidimensional preinvex stochastic processes and verified Hermite-Hadamard type inequality for these processes.

Keywords: Multidimensional stochastic process, preinvexity, mean-square integral, Hermite-Hadamard inequality.

2000 AMS Classification: Primary 26D15; Secondary 26A51.

1. INTRODUCTION

Convex functions are important and provide a base to build literature of mathematical inequalities. A function $f: I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in I$ and $\lambda \in [0,1]$. A classical inequality for convex functions is Hadamard's inequality, this is given as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

where $f: I \to \mathbb{R}$ is a convex function for all $a, b \in I$, $a < b$. This inequality is used to provide estimations of the mean value of a continuous convex function.

In the meanwhile, preinvexity is a generalization of convexity for functions. Therefore, Hermite-Hadamard type inequalities for preinvex functions were obtained by many researchers. For examples, Hanson [2], Ben-Isreal-Mond [3], Pini [4], Mohan-Neogy [5], Weir-Mond [6], Yang-Li [7], Noor [8,9], Mishra [10], etc. They have studied the basic properties of the preinvex functions and on their role in optimization, variational inequalities and equilibrium problems.

The classical Hermite-Hadamard type inequality for preinvex functions on $[a, a+\eta(b,a)]$ is as follows:

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

In terms of probability theory, this inequality gives a lower bound and an upper bound for the expectation value of a random variable X which distributed uniformly on $[a,b]$ [11].

In a sence, a stochastic process is a temporal parameterized family of random variables on a probability space [6]. In other words, let $(\Omega, \Im, P)$ be an arbitrary probability space. A function $X: \Omega \to \mathbb{R}$ is called a random variable, if it is $\Im$-measurable. Correspondingly, $X: I \subset \mathbb{R} \times \Omega \to \mathbb{R}$ is called a stochastic process, if $t \in I$ is considered of a time parameter of the random function $X(t, \omega)$ for all $\omega \in \Omega$. In this sense, researchers tackle many problems related convexity and inequality for stochastic processes privately [6]-[21].

Let us call up some basic notions related stochastic processes [6]. The process X is defined as

(i) continuous in probability on I, if $P - \lim_{t \to t_0} X(t,\cdot) = X(t_0,\cdot)$ for $t_0 \in I$, where $P\text{-}lim$ defined limit in probability.

(ii) mean-square continuous at $t_0 \in I$, if $\lim_{t \to t_0} E[X(t,\cdot) - X(t_0,\cdot)]^2 = 0$ where $E[X(t,\cdot)]$ is the expectation value of $X(t,\cdot)$.

Also, a stochastic process is mean-square continuous (differentiable) on I, if it is continuous (differentiable) on the interval I,

(iii) increasing (decreasing) $t < s$ $(t > s)$ if $X(t,\cdot) \leq X(s,\cdot)$ $\big(X(t,\cdot) \geq X(s,\cdot)\big)$ for all $t, s \in I$,

(v) monotonic if it is increasing or decreasing,

(vi) mean-square differentiable at a point $t \in I$, if there is $X'(t,\cdot): I \times \Omega \to \mathbb{R}$ such that

$$X'(t,\cdot) = P - \lim_{t \to t_0} \frac{X(t,\cdot) - X(t_0,\cdot)}{t - t_0}.$$

(vii) mean-square integrable on $[0, t] \subset I$, *iff*

$$\lim_{n \to \infty} E\left(\sum_{k=1}^{n} X(\Theta_k,\cdot).(t_k - t_{k-1}) - \eta(t,\cdot)\right)^2 = 0.$$

where $\Theta_k \in [t_{k-1}, t_k]$ such that $[t_{k-1}, t_k,], k = 1, \ldots, n$ for $0 = t_0 < t_1 < \cdots < t_n = t$ *is a partition of* $[0, t]$. Then almost everywhere, it can be sometimes showed with (a.e.),

$$\int_0^t X\,(u,\cdot)du = \eta(t,\cdot)$$

Concordantly, there are satisfactory evidence on similar results belong to stochastic processes. Nikodem [12] proposed convex stochastic process and its some properties in 1980. Some applications to stochastic convexity verified by Shaked et al. [13] in 1988. Jensen-convex, λ-convex stochastic processes were introduced by Skowronski [14] in 1992. The classical Hermite-Hadamard inequality to convex stochastic processes was extended by Kotrys [15] in 2012. Set et al. [16] obtained some inequalities for coordinated convex stochastic processes in 2015.

From 2018, Karahan et al [17,18,19,20] investigated multidimensional convex, s-convex, φ-convex, harmonically stochastic processes on n-coordinates and obtained Hermite-Hadamard type inequality for all of these processes.

Moreover, Okur et al. [21,22] defined some extensions for preinvex stochastic processes and obtained some inequalities for these processes in 2014. Finally, Okur [23] verified some generalizations of Hermite-Hadamard type inequalities for two-dimensional preinvex stochastic processes in 2019.

2. PRELIMINARIES

Let us recall some known results concerning invexty and preinvexity of stochastic processes.

Definition 2.1([21])**.** A set $I \subseteq \mathbb{R}^n$ is called invex with respect to the continuous function $\eta: I \times I \to \mathbb{R}^n$, if $t + t\eta(s,t) \in I, \forall t, s \in I, t \in [0,1]$. The invex set I is also called a η-connected set.

Definition 2.2([21])**.** Let $I \subseteq \mathbb{R}^n$ be invex with respect to $\eta: I \times I \to \mathbb{R}^n$, A stochastic process $X: I \times \Omega \to \mathbb{R}$ is called preinvex with respect to η, if

$$X(t + \lambda\eta(s,t),\cdot) \leq (1-\lambda)X(t,\cdot) + \lambda X(s,\cdot)$$

for all $t, s \in I$ and $\lambda \in [0,1]$ almost everywhere. If the above inequality is reversed, then X is said to be a pre-concave almost everywhere. Moreover, if we choose $\eta(t,s) = t - s$, then X is a convex stochastic process.

Mohan and Neogy [5] proved that an invex function is also preinvex under following Condition C.

Condition C: Let $I \subseteq \mathbb{R}^n$ be invex with respect to $\eta: I \times I \to \mathbb{R}^n$. It is told that the function η satisfies Condition C, if

$$\eta\big(s, s + \lambda\eta(t,s)\big) = -\lambda\eta(t,s);$$

$$\eta\big(t, s + \lambda\eta(t,s)\big) = (1-\lambda)\eta(t,s)$$

for all $t, s \in I$ and $\lambda \in [0,1]$. Additionally, note that from condition C, we have

$$\eta\big(s + \lambda_2\,\eta(s,t), s + \lambda_1\,\eta(t,s)\big) = (\lambda_2 - \lambda_1)\eta(t,s),$$

for all $t, s \in I$ and $\lambda, \lambda_1, \lambda_2 \in [0,1]$

Theorem 2.1([21])**.** Let $X: [u, u + \eta(v,u)] \times \Omega \to (0,\infty)$ *be a preinvex stochastic process and $u, v \in I^\circ$ with $u < u + \eta(v,u)$. If X is mean-square integrable on $[u, u + \eta(v,u)]$, then the following inequalitiy holds almost everywhere*

$$X\left(\frac{2u + \eta(v,u)}{2},\cdot\right) \leq \frac{1}{\eta(v,u)} \int_{u}^{u+\eta(v,u)} X(t,\cdot)dt \leq \frac{X(u,\cdot) + X(v,\cdot)}{2}.$$

3. MAIN RESULTS

The main goal of this section is to present Hermite-Hadamard type inequalities for preinvex stochastic processes with respect to η on n-coordinates.

Let u_i, v_i be real numbers such that $u_i < v_i$ for $i = 1,2,\ldots,n,\ n \geq 2$. We consider n-dimensional interval $\Delta^n = \prod_{i=1}^n [u_i, v_i] \subseteq [0,\infty)^n$.

Firstly, we give definition of preinvexity for stochastic processes with respect to η on n-coordinates:

Definition 3.1. *A stochastic process $X: \Delta^n \times \Omega \to \mathbb{R}$ is called preinvex with respect to η on n-coordinates if the stochastic processes*

$$X_{t_n}^i(t,\cdot) := X\big((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n),\cdot\big) \ \ (a.e.)$$

are preinvex with respect to η on $[u_i, v_i]$ for $i = 1,2, \dots, n$ almost everywhere.

Definition 3.2. *A stochastic process $X: \Delta^n \times \Omega \to \mathbb{R}$ is said to be preinvex with respect to η on Δ^n if the following inequality holds almost everywhere*

$$X\Big(\big(\boldsymbol{t} + \lambda\eta(\boldsymbol{s},\boldsymbol{t})\big),\cdot\Big) \le \lambda X(\boldsymbol{s},\cdot) + (1-\lambda)X(\boldsymbol{t},\cdot)$$

for all $\boldsymbol{t} = (t_1, t_2, \dots, t_n), \boldsymbol{s} = (s_1, s_2, \dots, s_n) \in \Delta^n$ and $\lambda \in [0,1]$. If the above inequality is reversed then X is said to be preconcave with respect to η on Δ^n.

Lemma 3.1. *Every multidimensional preinvex stochastic process with respect to η on Δ^n is preinvex with respect to η on n-coordinates, but converse is not true.*

Proof. Let $X: \Delta^n \times \Omega \to \mathbb{R}$ be a preinvex stochastic process with respect to η on Δ^n. Consider $X_{t_n}^i : [u_i, v_i] \times \Omega \to \mathbb{R}$ defined by

$$X_{t_n}^i(t,\cdot) := X\big((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n),\cdot\big), \qquad t \in [u_i, v_i].$$

Now for $t, s \in [u_i, v_i]$ and $\lambda \in [0,1]$ almost everywhere

$$X_{t_n}^i\Big(\big(t + \lambda\eta(s,t)\big),\cdot\Big) := X\Big(\big(t_1, \dots, t_{i-1}, \big(t + \lambda\eta(s,t)\big), t_{i+1}, \dots, t_n\big),\cdot\Big)$$

$$\le \lambda X\big((t_1, \dots, t_{i-1}, s, t_{i+1}, \dots, t_n),\cdot\big) + (1-\lambda)X\big((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n),\cdot\big)$$

$$= \lambda X_{t_n}^i(s,\cdot) + (1-\lambda)X_{t_n}^i(t,\cdot)$$

which implies $X_{t_n}^i$ is preinvex with respect to η on $[u_i, v_i]$, that is, X is preinvex with respect to η on n-coordinates. For converse we give the following counter example:

Example 3.1. Let us consider a stochastic process $X: [0,1]^n \times \Omega \to \mathbb{R}$ defined as

$$X_{t_n}^i(t,\cdot) := X\big((t_1, \dots, t_{i-1}, t, t_{i+1}, \dots, t_n),\cdot\big) = \ t_1\ t_2\ \dots\ t_n.$$

for $t \in [u_i, v_i]$. Moreover let be $\eta: [0,1]^n \to [0,\infty)$, $\eta(\boldsymbol{s},\boldsymbol{t}) = \boldsymbol{s} - \boldsymbol{t}$, for all $\boldsymbol{t} = (t_1, t_2, \dots, t_n), \boldsymbol{s} = (s_1, s_2, \dots, s_n) \in \Delta^n$. Then X is not preinvex with respect to η on $[0,1]^n$.

Indeed, let us assume that X is preinvex with respect to η on $[0,1]^n$, then we can write

$$X\Big(\big(\boldsymbol{t} + \lambda\eta(\boldsymbol{s},\boldsymbol{t})\big),\cdot\Big) \le \lambda X(\boldsymbol{s},\cdot) + (1-\lambda)X(\boldsymbol{t},\cdot) \ \ (a.e.)$$

for all $\boldsymbol{t} = (t_1, t_2, \dots, t_n),\ \boldsymbol{s} = (s_1, s_2, \dots, s_n) \in \Delta^n$ *and* $\lambda \in [0,1]$.

But for $\boldsymbol{t} = (1,1,, \dots,1, 0),\ \boldsymbol{s} = (0,1, \dots,1) \in [0,1]^n$, we have

$$X\left(\left(\boldsymbol{t}+\lambda\eta(\boldsymbol{s},\boldsymbol{t})\right),\cdot\right)=X\left(\left((1,1,\dots,1,0)+\lambda\left((0,1,\dots,1)-(1,1,\dots 1,0)\right)\right),\cdot\right)$$

$$=X\left(\left((1,1,\dots,1,0)+\lambda(-1,0,\dots,0,1)\right),\cdot\right)$$

$$=X\left((1-\lambda,1,\dots,\mathbf{1},\lambda),\cdot\right)=\lambda(1-\lambda)$$

and since $X(\boldsymbol{t},\cdot)=0$ and $X(\boldsymbol{s},\cdot)=0$ for $\boldsymbol{t}=(1,1,,\dots,1,0),\boldsymbol{s}=(0,1,\dots,1)\in[0,1]^n$, then

$$\lambda X(\boldsymbol{s},\cdot)+(1-\lambda)X(\boldsymbol{t},\cdot)=\lambda.0+(1-\lambda).0=0.$$

This gives

$$X\left(\left(\boldsymbol{t}+\lambda\eta(\boldsymbol{s},\boldsymbol{t})\right),\cdot\right)>\lambda X(\boldsymbol{s},\cdot)+(1-\lambda)X(\boldsymbol{t},\cdot)$$

for all $\lambda\in[0,1]$, that is, X is not preinvex with respect to η on $[0,1]^n$. That case is contradiction with preinvexity of X.

From here on out, let us assume that the n-dimensional interval is $\Lambda^n=\prod_{i=1}^n[u_i,u_i+\eta(v_i,u_i)]\subseteq[0,\infty)^n$.

Remark 3.1. *If $X:\Lambda^n\times\Omega\to\mathbb{R}_+$ is a preinvex stochastic process with respect to η on n-coordinates, then $X_{t_n}^i:[u_i,u_i+\eta(v_i,u_i)]\times\Omega\to\mathbb{R}$ is preinvex with respect to η on $[u_i,u_i+\eta(v_i,u_i)]$ for each $i=1,2,\dots,n$. From Hermite-Hadamard inequality,*

$$X_{t_n}^i\left(\frac{2u_i+\eta(v_i,u_i)}{2},\cdot\right)\le\frac{1}{\eta(v_i,u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)}X_{t_n}^i\,(t_i,\cdot)dt_i\le\frac{X_{t_n}^i\,(u_\mathrm{i},\cdot)+X_{t_n}^i\,(v_\mathrm{i},\cdot)}{2}.\quad(1)$$

Theorem 3.1. *Let $X:\Lambda^n\times\Omega\to\mathbb{R}_+$ be preinvex stochastic process with respect to η on n-coordinates. If X is mean-square integrable on $[u_i,u_i+\eta(v_i,u_i)]$* for each $i=1,2,\dots,n$, *then we obtain almost everywhere*

$$\sum_{i=1}^n\frac{1}{\eta(v_i,u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)}X_{t_n}^{i+1}\left(\frac{2u_{i+1}+\eta(v_{i+1},u_{i+1})}{2},\cdot\right)dt_i$$

$$\le\sum_{i=1}^n\frac{1}{\eta(v_i,u_i)\eta(v_{i+1},u_{i+1})}\int_{u_i}^{u_i+\eta(v_i,u_i)}\int_{u_{i+1}}^{u_{i+1}+\eta(v_{i+1},u_{i+1})}X(t,\cdot)\;dt_{i+1}\,dt_i$$

$$\le\sum_{i=1}^n\frac{1}{2\eta(v_i,u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)}\left[X_{t_n}^{i+1}(u_{i+1},\cdot)+X_{t_n}^{i+1}(v_{i+1},\cdot)\right]dt_i\;.\qquad(2)$$

Proof. By applying Hermite-Hadamard's inequality for the preinvex stochastic process $X_{t_n}^{i+1}$ on interval $[u_{i+1},u_{i+1}+\eta(v_{i+1},u_{i+1})\,]$, we have almost everywhere

$$X_{t_n}^{i+1}\left(\frac{2u_{i+1}+\eta(v_{i+1},u_{i+1})}{2},\cdot\right)\leq\frac{1}{\eta(v_{i+1},u_{i+1})}\int_{u_{i+1}}^{u_{i+1}+\eta(v_{i+1},u_{i+1})}X_{t_n}^{i+1}(t_{i+1},\cdot)dt_{i+1}$$

$$\leq\frac{X_{t_n}^{i+1}(u_{i+1},\cdot)+X_{t_n}^{i+1}(v_{i+1},\cdot)}{2}.$$

Since X is mean-square integrable on $[u_i,u_i+\eta(v_i,u_i)]$ for each $i=1,2,\dots,n$, we get almost everywhere

$$\frac{1}{\eta(v_i,u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)}X_{t_n}^{i+1}\left(\frac{u_{i+1}+v_{i+1}}{2},\cdot\right)dt_i$$

$$\leq\frac{1}{\eta(v_i,u_i)\eta(v_{i+1},u_{i+1})}\int_{u_i}^{u_i+\eta(v_i,u_i)}\int_{u_{i+1}}^{u_{i+1}+\eta(v_{i+1},u_{i+1})}X_{t_n}^{i+1}(t_{i+1},\cdot)\ dt_{i+1}\,dt_i$$

$$\leq\frac{1}{2\eta(v_i,u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)}\left(X_{t_n}^{i+1}(u_{i+1},\cdot)+X_{t_n}^{i+1}(v_{i+1},\cdot)\right)dt_i\,.$$

Taking summation from 1 to n we get (2).

Theorem 3.2. *Let $X:\Lambda^n\times\Omega\to\mathbb{R}_+$ be preinvex stochastic process with respect to η on n-coordinates. If X is mean-square integrable on* $[u_i,u_i+\eta(v_i,u_i)]$ for each $i=1,2,\dots,n$, *then for $\boldsymbol{u},\boldsymbol{v}\in\Lambda^n$ we obtain almost everywhere*

$$\sum_{i=1}^{n}\frac{1}{2\eta(v_i,u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)}\left(X_{u_n}^{i}(t_i,\cdot)+X_{v_n}^{i}(t_i,\cdot)\right)dt_i$$

$$\leq\frac{n}{2}[X(\boldsymbol{u},\cdot)+X(\boldsymbol{v},\cdot)]+\frac{1}{2}\sum_{i=1}^{n}\left[X_{u_n}^{i}(v_i,\cdot)+X_{v_n}^{i}(u_i,\cdot)\right]. \tag{3}$$

Proof. Since $X:\Lambda^n\times\Omega\to\mathbb{R}_+$ is a preinvex stochastic process with respect to η on n-coordinates, then for $X_{t_n}^{i}:[u_i,u_i+\eta(v_i,u_i)]\times\Omega\to\mathbb{R}_+$ is a preinvex stochastic process with respect to η on $[u_i,u_i+\eta(v_i,u_i)]$ for each $i=1,2,\dots,n$, we have almost everywhere

$$\frac{1}{\eta(v_i,u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)}X_{u_n}^{i}(t_i,\cdot)dt_i\ \leq\frac{X(\boldsymbol{u},\cdot)+X_{u_n}^{i}(v_i,\cdot)}{2}$$

$$\frac{1}{\eta(v_i,u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)}X_{v_n}^{i}(t_i,\cdot)dt\ \leq\frac{X_{v_n}^{i}(u_i,\cdot)+X(\boldsymbol{v},\cdot)}{2}$$

Adding above two inequalities we have

$$\frac{1}{\eta(v_i,u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)}\left[X_{u_n}^{i}(t_i,\cdot)+X_{v_n}^{i}(t_i,\cdot)\right]dt_i\ \leq\frac{X(\boldsymbol{u},\cdot)+X(\boldsymbol{v},\cdot)+X_{u_n}^{i}(v_i,\cdot)+X_{v_n}^{i}(u_i,\cdot)}{2},$$

Taking summation from 1 to n the above inequalities we get (3).

Corollary 3.1. *Let $X\colon \Lambda^2 \times \Omega \to \mathbb{R}_+$ be preinvex stochastic process with respect to η on 2-coordinates. If X is mean-square integrable on Λ^2, then we get almost everywhere*

$$X\left(\left(\frac{2u_1+\eta(v_1,u_1)}{2},\frac{2u_2+\eta(v_2,u_2)}{2}\right),\cdot\right)$$

$$\leq \frac{1}{2}\left[\frac{1}{\eta(v_1,u_1)}\int_{u_1}^{u_1+\eta(v_1,u_1)} X\left(\left(t_1,\frac{2u_2+\eta(v_2,u_2)}{2}\right),\cdot\right)dt_1 + \frac{1}{\eta(v_2,u_2)}\int_{u_2}^{u_2+\eta(v_2,u_2)} X\left(\left(\frac{2u_1+\eta(v_1,u_1)}{2},t_2\right),\cdot\right)dt_2\right]$$

$$\leq \frac{1}{\eta(v_1,u_1)\eta(v_2,u_2)}\int_{u_1}^{u_1+\eta(v_1,u_1)}\int_{u_2}^{u_2+\eta(v_2,u_2)} X\big((t_1,t_2),\cdot\big)dt_2dt_1$$

$$\leq \frac{1}{4\eta(v_1,u_1)}\int_{u_1}^{u_1+\eta(v_1,u_1)}\big[X\big((t_1,u_2),\cdot\big)+X\big((t_1,v_2),\cdot\big)\big]dt_1$$

$$+\frac{1}{4\eta(v_2,u_2)}\int_{u_2}^{u_2+\eta(v_2,u_2)}\big[X\big((u_1,t_2),\cdot\big)+X\big((v_1,t_2),\cdot\big)\big]dt_2$$

$$\leq \frac{X(u_1,u_2),\cdot + X\big((u_1,v_2),\cdot\big)+X\big((v_1,u_2),\cdot\big)+X\big((v_1,v_2),\cdot\big)}{4}.$$

Theorem 3.3. *If $X\colon \Lambda^n \times \Omega \to \mathbb{R}_+$ is preinvex stochastic process on n-coordinates, then $X_{t_n}^k : [u_k, u_k+\eta(v_k,u_k)] \times \Omega \to \mathbb{R}_+$ is preinvex on $[u_k, u_k+\eta(v_k,u_k)]$ for each $k = 1,2,\dots,n$. From Hermite-Hadamard inequality, we have almost everywhere*

$$\sum_{k=1}^{n-1} X\left(\left(t_1,\dots,t_{k-1},\frac{2u_k+\eta(v_k,u_k)}{2},\frac{2u_{k+1}+\eta(v_{k+1},u_{k+1})}{2},\dots,t_n\right),\cdot\right)$$

$$\leq \sum_{k=1}^{n-1}\frac{1}{\eta(v_k,u_k)}\int_{u_k}^{u_k+\eta(v_k,u_k)} X_{t_n}^k\left(\frac{2u_k+\eta(v_k,u_k)}{2},\cdot\right)dt_k$$

$$\leq \sum_{k=1}^{n-1}\frac{1}{\eta(v_k,u_k)\eta(v_{k+1},u_{k+1})}\int_{u_k}^{u_k+\eta(v_k,u_k)}\int_{u_{k+1}}^{u_{k+1}+\eta(v_{k+1},u_{k+1})} X_{t_n}^{k+1}(t_{k+1},\cdot)dt_{k+1}dt_k$$

$$\leq \sum_{k=1}^{n-1}\frac{1}{2\eta(v_k,u_k)}\int_{u_k}^{u_k+\eta(v_k,u_k)}\Big(X_{t_n}^k(u_{k+1},\cdot)+X_{t_n}^k(v_{k+1},\cdot)\Big)dt_k$$

$$\leq \frac{1}{4}\sum_{k=1}^{n-1}\left[\begin{aligned}&X\big((t_1,\dots,t_{k-1},u_k,u_{k+1},\dots,t_n),\cdot\big)+X\big((t_1,\dots,t_{k-1},v_k,u_{k+1},\dots,t_n),\cdot\big)\\&+X\big((t_1,\dots,t_{k-1},u_k,v_{k+1},\dots,t_n),\cdot\big)+X\big((t_1,\dots,t_{k-1},v_k,v_{k+1},\dots,t_n),\cdot\big)\end{aligned}\right]. \quad (4)$$

Proof. Since $X:\Lambda^n \times \Omega \to \mathbb{R}_+$ is preinvex stochastic on n-coordinates, there for $X_{t_n}^i:[u_i, u_i + \eta(v_i, u_i)] \times \Omega \to \mathbb{R}_+$ is preinvex stochastic process on $[u_i, u_i + \eta(v_i, u_i)]$ for each $i = 1,2,\dots,n$, we have on $[u_{i+1}, v_{i+1}\ ]$ almost everywhere

$$X_{t_n}^{i+1}\left(\frac{2u_{i+1} + \eta(v_{i+1}, u_{i+1})}{2}, \cdot\right) \leq \frac{1}{\eta(v_{i+1}, u_{i+1})}\int_{u_{i+1}}^{u_{i+1}+\eta(v_{i+1},u_{i+1})} X_{t_n}^{i+1}(t_{i+1}, \cdot)dt_{i+1}$$

$$\leq \frac{X_{t_n}^{i+1}(u_{i+1}, \cdot) + X_{t_n}^{i+1}(v_{i+1}, \cdot)}{2}.$$

All of sides of the above inequalities by integrating over $[u_i, u_i + \eta(v_i, u_i)]$

$$\frac{1}{\eta(v_i, u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)} X_{t_n}^{i+1}\left(\frac{2u_{i+1} + \eta(v_{i+1}, u_{i+1})}{2}, \cdot\right) dt_i$$

$$\leq \frac{1}{\eta(v_i, u_i)\eta(v_{i+1}, u_{i+1})}\int_{u_i}^{u_i+\eta(v_i,u_i)} \int_{u_{i+1}}^{u_{i+1}+\eta(v_{i+1},u_{i+1})} X_{t_n}^{i+1}(t_{i+1}, \cdot)dt_{i+1}dt_i$$

$$\leq \frac{1}{2\eta(v_i, u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)} \left(X_{t_n}^{i+1}(u_{i+1}, \cdot) + X_{t_n}^{i+1}(v_{i+1}, \cdot)\right) dt_i. \qquad (5)$$

Again applying the Hermite-Hadamard inequality

$$X\left(\left(t_1, \dots, \frac{2u_i + \eta(v_i, u_i)}{2}, \frac{2u_{i+1} + \eta(v_{i+1}, u_{i+1})}{2}, \dots, t_n\right), \cdot\right)$$

$$\leq \frac{1}{\eta(v_i, u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)} X_{t_n}^{i+1}\left(\frac{2u_{i+1} + \eta(v_{i+1}, u_{i+1})}{2}, \cdot\right) dt_i \qquad (6)$$

for each $i \in \{1,2,\dots,n-1\}$ and also

$$\frac{1}{2\eta(v_i, u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)} \left(X_{t_n}^{i+1}(u_{i+1}, \cdot) + X_{t_n}^{i+1}(v_{i+1}, \cdot)\right) dt_i$$

$$= \frac{1}{2}\left[\frac{1}{\eta(v_i, u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)} X_{t_n}^{i+1}(u_{i+1}, \cdot)dt_i + \frac{1}{\eta(v_i, u_i)}\int_{u_i}^{u_i+\eta(v_i,u_i)} X_{t_n}^{i+1}(v_{i+1}, \cdot)dt_i\right]$$

$$\leq \frac{1}{2}\left[\begin{array}{c} \frac{X\left((t_1, \dots, t_{i-1}, u_i, u_{i+1}, \dots, t_n), \cdot\right) + X\left((t_1, \dots, t_{i-1}, v_i, u_{i+1}, \dots, t_n), \cdot\right)}{2} \\ + \frac{X\left((t_1, \dots, t_{i-1}, u_i, v_{i+1}, \dots, t_n), \cdot\right) + X\left((t_1, \dots, t_{i-1}, v_i, v_{i+1}, \dots, t_n), \cdot\right)}{2} \end{array}\right]$$

$$= \frac{1}{4}\left[\begin{array}{c} X\left((t_1, \dots, t_{i-1}, u_i, u_{i+1}, \dots, t_n), \cdot\right) + X\left((t_1, \dots, t_{i-1}, v_i, u_{i+1}, \dots, t_n), \cdot\right) \\ + X\left((t_1, \dots, t_{i-1}, u_i, v_{i+1}, \dots, t_n), \cdot\right) + X\left((t_1, \dots, t_{i-1}, v_i, v_{i+1}, \dots, t_n), \cdot\right) \end{array}\right] \qquad (7)$$

for each $i \in \{1,2,\dots,n-1\}$. Using the inequalities (6) and (7) in (5) and taking summation from 1 to $n-1$, we have (4).

Theorem 3.4. *Let $X: \Lambda^n \times \Omega \to \mathbb{R}_+$ be a preinvex stochastic process and be integrated in mean-square on Λ^n. Then almost everywhere*

$$X\left(\left(\frac{2u_1+\eta(v_1,u_1)}{2},\dots,\frac{2u_{n-1}+\eta(v_{n-1},u_{n-1})}{2},\frac{2u_n+\eta(v_n,u_n)}{2}\right),\cdot\right)$$

$$\leq \left(\prod_{i=1}^{n}\frac{1}{\eta(v_i,u_i)}\right)\int_{u_1}^{u_1+\eta(v_1,u_1)}\dots\int_{u_n}^{u_n+\eta(v_n,u_n)} X_{t_n}^n\,(t_n,\cdot)dt_n \dots dt_1$$

$$\leq \frac{1}{2^n}\sum_{\delta\in l_i(n)} X(\delta u+(1-\delta)v,\cdot), \tag{8}$$

where $l_i(n) \coloneqq \{\delta \in \mathbb{N}_0^n: \delta \leq 1, |\delta| = n+1-i, i = 1,\dots,n+1\}$, $|\delta| \coloneqq \delta_1+\dots+\delta_n \in \mathbb{N}$; $\delta u := (\delta_1 u_1,\dots,\delta_n u_n) \in \mathbb{N}_0^n$ for $u, v \in \Delta^n$.

Proof. Let $\omega_n \coloneqq u_n+\eta(v_n,u_n)$. Then using (1), we get the following inequality for $X_{t_n}^n$ almost everywhere

$$X_{t_n}^n\left(\frac{u_n+\omega_n}{2},\cdot\right) \leq \frac{1}{\omega_n-u_n}\int_{u_n}^{\omega_n} X_{t_n}^n\,(t_n,\cdot)dt_n \leq \frac{X_{t_n}^n\,(u_n,\cdot)+X_{t_n}^n\,(v_n,\cdot)}{2}. \tag{9}$$

By integrating on $[u_{n-1},\omega_{n-1}]$, we get

$$\frac{1}{\omega_{n-1}-u_{n-1}}\int_{u_{n-1}}^{\omega_{n-1}} X_{t_n}^n\left(\frac{u_n+\omega_n}{2},\cdot\right)dt_{n-1}$$

$$\leq \frac{1}{(\omega_{n-1}-u_{n-1})(\omega_n-u_n)}\int_{u_{n-1}}^{\omega_{n-1}}\int_{u_n}^{\omega_n} X_{t_n}^n\,(t_n,\cdot)dt_n dt_{n-1}$$

$$\leq \frac{1}{\omega_{n-1}-u_{n-1}}\int_{u_{n-1}}^{\omega_{n-1}}\frac{X_{t_n}^n\,(u_n,\cdot)+X_{t_n}^n\,(v_n,\cdot)}{2}dt_{n-1}. \tag{10}$$

From (5), (6), respectively

$$X\left(\left(t_1,\dots,\frac{u_{n-1}+\omega_{n-1}}{2},\frac{u_n+\omega_n}{2}\right),\cdot\right) \leq \frac{1}{\omega_{n-1}-u_{n-1}}\int_{u_{n-1}}^{\omega_{n-1}} X_{t_n}^n\left(\frac{u_n+\omega_n}{2},\cdot\right)dt_{n-1}, \tag{11}$$

$$\leq \frac{1}{\omega_{n-1}-u_{n-1}}\int_{u_{n-1}}^{\omega_{n-1}}\frac{X_{t_n}^n\,(u_n,\cdot)+X_{t_n}^n\,(v_n,\cdot)}{2}dt_{n-1}$$

$$= \frac{1}{2(\omega_{n-1}-u_{n-1})}\int_{u_{n-1}}^{\omega_{n-1}} X_{t_n}^n\,(u_n,\cdot)dt_{n-1} + \frac{1}{2(\omega_{n-1}-u_{n-1})}\int_{u_{n-1}}^{\omega_{n-1}} X_{t_n}^n\,(v_n,\cdot)dt_{n-1}$$

$$\leq \frac{1}{2^2}\begin{bmatrix} X\big((t_1,\dots,u_{n-1},u_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\dots,u_{n-1},v_n),\cdot\big)+X\big((t_1,\dots,v_{n-1},v_n),\cdot\big)\end{bmatrix}. \tag{12}$$

From (10)-(12)

$$X\left(\left(t_1,\ldots,\frac{u_{n-1}+\omega_{n-1}}{2},\frac{u_n+\omega_n}{2}\right),\cdot\right)$$

$$\leq \frac{1}{(\omega_{n-1}-u_{n-1})(\omega_n-u_n)}\int_{u_{n-1}}^{\omega_{n-1}}\int_{u_n}^{\omega_n} X_{t_n}^n\,(t_n,\cdot)dt_n dt_{n-1}$$

$$\leq \frac{1}{2^2}\begin{bmatrix} X\big((t_1,\ldots,u_{n-1},u_n),\cdot\big)+X\big((t_1,\ldots,v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\ldots,u_{n-1},v_n),\cdot\big)+X\big((t_1,\ldots,v_{n-1},v_n),\cdot\big)\end{bmatrix}. \tag{13}$$

Integrating on $[u_{n-2},\omega_{n-2}]$

$$\frac{1}{\omega_{n-1}-u_{n-1}}\int_{u_{n-2}}^{\omega_{n-2}} X\left(\left(t_1,\ldots,\frac{u_{n-1}+\omega_{n-1}}{2},\frac{u_n+\omega_n}{2}\right),\cdot\right)dt_{n-2}$$

$$\leq \left(\prod_{i=n-2}^{n}\frac{1}{\omega_i-u_i}\right)\int_{u_{n-2}}^{\omega_{n-2}}\int_{u_{n-1}}^{\omega_{n-1}}\int_{u_n}^{\omega_n} X_{t_n}^n\,(t_n,\cdot)dt_n dt_{n-1}\,dt_{n-2}$$

$$\leq \frac{1}{(\omega_{n-2}-u_{n-2})}\int_{u_{n-2}}^{\omega_{n-2}}\frac{1}{2^2}\begin{bmatrix} X\big((t_1,\ldots,u_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\ldots,v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\ldots,u_{n-1},v_n),\cdot\big) \\ +X\big((t_1,\ldots,v_{n-1},v_n),\cdot\big)\end{bmatrix}dt_{n-2}. \tag{14}$$

Similarly

$$X\left(\left(t_1,\ldots,\frac{u_{n-2}+\omega_{n-2}}{2},\frac{u_{n-1}+\omega_{n-1}}{2},\frac{u_n+\omega_n}{2}\right),\cdot\right)$$

$$\leq \frac{1}{\omega_{n-2}-u_{n-2}}\int_{u_{n-2}}^{\omega_{n-2}} X\left(\left(t_1,\ldots,\frac{u_{n-1}+\omega_{n-1}}{2},\frac{u_n+\omega_n}{2}\right),\cdot\right)dt_{n-2}, \tag{15}$$

$$\leq \frac{1}{\omega_{n-2}-u_{n-2}}\int_{u_{n-2}}^{\omega_{n-2}}\frac{1}{2^2}\begin{bmatrix} X\big((t_1,\ldots,u_{n-1},u_n),\cdot\big)+X\big((t_1,\ldots,v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\ldots,u_{n-1},v_n),\cdot\big)+X\big((t_1,\ldots,v_{n-1},v_n),\cdot\big)\end{bmatrix}dt_{n-2}$$

$$\leq \frac{1}{2^3}\begin{bmatrix} X\big((t_1,\ldots,u_{n-2},u_{n-1},u_n),\cdot\big)+X\big((t_1,\ldots,v_{n-2},u_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\ldots,u_{n-2},v_{n-1},u_n),\cdot\big)+X\big((t_1,\ldots,v_{n-2},v_{n-1},u_n),\cdot\big) \\ +X\big((t_1,\ldots,u_{n-2},u_{n-1},v_n),\cdot\big)+X\big((t_1,\ldots,v_{n-2},u_{n-1},v_n),\cdot\big) \\ +X\big((t_1,\ldots,u_{n-2},v_{n-1},v_n),\cdot\big)+X\big((t_1,\ldots,v_{n-2},v_{n-1},v_n),\cdot\big)\end{bmatrix}. \tag{16}$$

From (14)-(16)

$$\frac{1}{\omega_{n-2}-u_{n-2}}\int_{u_{n-2}}^{\omega_{n-2}} X\left(\left(t_1,\ldots,\frac{u_{n-1}+\omega_{n-1}}{2},\frac{u_n+\omega_n}{2}\right),\cdot\right)dt_{n-2}$$

$$\leq \left(\prod_{i=n-2}^{n}\frac{1}{\omega_i-u_i}\right)\int_{u_{n-2}}^{\omega_{n-2}}\int_{u_{n-1}}^{\omega_{n-1}}\int_{u_n}^{\omega_n} X_{t_n}^n\,(t_n,\cdot)dt_n\,dt_{n-1}dt_{n-2}$$

$$\leq \frac{1}{2^3}\begin{bmatrix} X((t_1,\dots,u_{n-2},u_{n-1},u_n),\cdot)+X((t_1,\dots,v_{n-2},u_{n-1},u_n),\cdot) \\ +X((t_1,\dots,u_{n-2},v_{n-1},u_n),\cdot)+X((t_1,\dots,v_{n-2},v_{n-1},u_n),\cdot) \\ +X((t_1,\dots,u_{n-2},u_{n-1},v_n),\cdot)+X((t_1,\dots,v_{n-2},u_{n-1},v_n),\cdot) \\ +X((t_1,\dots,u_{n-2},v_{n-1},v_n),\cdot)+X((t_1,\dots,v_{n-2},v_{n-1},v_n),\cdot) \end{bmatrix}.$$

Using inductive method for $n = k - 1$, we obtain

$$X\left(\left(\frac{u_1+\omega_1}{2},\dots,\frac{u_{k-1}+\omega_{k-1}}{2}\right),\cdot\right)$$

$$\leq \left(\prod_{i=1}^{k-1}\frac{1}{\omega_i-u_i}\right)\int_{u_1}^{\omega_1}\dots\int_{u_{k-1}}^{\omega_{k-1}} X_{t_n}^n(t_n,\cdot)dt_{k-1}\dots dt_1 \leq \frac{1}{2^{k-1}}\sum_{\delta\in l_i(k-1)} X(\delta u+(1-\delta)v,\cdot)$$

where $l_i(k-1) \coloneqq \{\delta \in \mathbb{N}_0^n: \delta \leq 1, |\delta| = k-i;\ i = 1,\dots,k, k \in \mathbb{N}\}$, $|\delta| \coloneqq \delta_1+\cdots+\delta_n \in \mathbb{N}$; $\delta u := (\delta_1 u_1,\dots,\delta_n u_n) \in \mathbb{N}_0^n$ for $u, v \in \Delta^n$.

Consequently, for $n = k$, we get (8) by using $\omega_n \coloneqq u_n + \eta(v_n, u_n)$.

Example 1. *Let $X: \Lambda^3 \times \Omega \to \mathbb{R}_+$ be a preinvex stochastic process and be integrated in mean-square on Λ^3. Then almost everywhere*

$$X\left(\left(\frac{2u_1+\eta(v_1,u_1)}{2},\frac{2u_2+\eta(v_2,u_2)}{2},\frac{2u_3+\eta(v_3,u_3)}{2}\right),\cdot\right)$$

$$\leq \frac{1}{\eta(v_1,u_1)\eta(v_2,u_2)\eta(v_3,u_3)}\int_{u_1}^{u_1+\eta(v_1,u_1)}\int_{u_2}^{u_2+\eta(v_2,u_2)}\int_{u_3}^{u_3+\eta(v_3,u_3)} X((t_1,t_2,t_3),\cdot)dt_3dt_2dt_1$$

$$\leq \frac{1}{2^3}\begin{bmatrix} X((u_1,u_2,u_3),\cdot)+X((v_1,u_2,u_3),\cdot) \\ +X((u_1,v_2,u_3),\cdot)+X((v_1,v_2,u_3),\cdot) \\ +X((u_1,u_2,v_3),\cdot)+X((v_1,u_2,v_3),\cdot) \\ +X((u_1,v_2,v_3),\cdot)+X((v_1,v_2,v_3),\cdot) \end{bmatrix}.$$

Proof. According to Theorem 2 for n = 3, we get $X_{t_3}^3(t_3,\cdot) \coloneqq X((t_1,t_2,t_3),\cdot)$ and $l_i(3) \coloneqq \{\kappa \in \mathbb{N}_0^3: \kappa \leq 1, |\kappa| = 4-i\}, i = 1,2,3,4$. Then

$$l_1(3) = \{(1,1,1)\};\ l_2(3) = \{(0,1,1), (1,0,1), (1,1,0)\},$$

$$l_3(3) = \{(0,0,1), (0,1,0), (1,0,0),\}; l_4(3) = \{(0,0,0)\},$$

and for $u = (u_1,u_2,u_3)$, $v = (v_1,v_2,v_3) \in \Delta^3$

$$\sum_{\delta\in l_1(3)} X(\delta u+(1-\delta)v,\cdot)$$

$$= X((1,1,1)(u_1,u_2,u_3)+[(1,1,1)-(1,1,1)](v_1,v_2,v_3),\cdot) = X((u_1,u_2,u_3),\cdot);$$

$$\sum_{\delta\in l_2(3)} X(\delta u+(1-\delta)v,\cdot)= X\big((0,1,1)(u_1,u_2,u_3)+[(1,1,1)-(0,1,1)](v_1,v_2,v_3),\cdot\big)$$

$$+X\big((1,0,1)(u_1,u_2,u_3)+[(1,1,1)-(1,0,1)](v_1,v_2,v_3),\cdot\big)$$

$$+X\big((1,1,0)(u_1,u_2,u_3)+[(1,1,1)-(1,1,0)](v_1,v_2,v_3),\cdot\big)$$

$$=X\big((u_1,v_2,v_3),\cdot\big)+X\big((u_1,v_2,u_3),\cdot\big)+X\big((u_1,u_2,v_3),\cdot\big);$$

So,

$$\sum_{\delta\in l_3(3)} X(\delta u+(1-\delta)v,\cdot)=X\big((v_1,v_2,u_3),\cdot\big)+X\big((v_1,u_2,v_3),\cdot\big)+X\big((u_1,v_2,v_3),\cdot\big);$$

$$\sum_{\delta\in l_4(3)} X(\delta u+(1-\delta)v,\cdot)=X\big((v_1,v_2,v_3),\cdot\big).$$

Thus

$$\sum_{\delta\in l_i(3)} X(\delta u+(1-\delta)v,\cdot)=X\big((u_1,u_2,u_3),\cdot\big)$$

$$+X\big((u_1,v_2,v_3),\cdot\big)+X\big((u_1,v_2,u_3),\cdot\big)+X\big((u_1,u_2,v_3),\cdot\big)$$

$$+X\big((v_1,v_2,u_3),\cdot\big)+X\big((v_1,u_2,v_3),\cdot\big)+X\big((u_1,v_2,v_3),\cdot\big)+X\big((v_1,v_2,v_3),\cdot\big).$$

Using all of the above equalities in (13), we obtain the desired result in this example.

ACKNOWLEDGEMENTS

The author would like to thank to Assoc. Prof. İmdat İŞCAN, Giresun University, Department of Mathematics, owing to his ideas.

4. CONCLUSION

In this paper, we obtained some new Hermite-Hadamard type inequalities for preinvex stochastic processes with respect to η on n-coordinates. As special cases, one can obtain several new and correct versions of the previously known results for various classes of these stochastic processes. Applying some type of inequalities for stochastic processes is another promising direction for future research.

REFERENCES

1. J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d.une function considerée par Riemann, J. Math Pures Appl., 58 (1893), 171.215.
2. M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.
3. A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1986), No.1, 1-9.
4. R. Pini, Invexity and generalized Convexity, Optimization 22 (1991) 513-525
5. S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995), 901.908.
6. T. Weir, and B. Mond, Preinvex functions in multiple bijective optimization, Journal of Mathematical Analysis and Applications, 136 (1998) 29-38.
7. X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001), 229-241.
8. M. A. Noor, Invex equilibrium problems, J. Math. Anal. Appl., 302 (2005), 463.475.
9. M. A. Noor, On Hadamard integral inequalities involving two log-preinvex functions, J. Inequal. Pure Appl. Math., 8(2007), No. 3, 1
10. Mishra, S. K., and Giorgi, G. 2008. Invexity and optimization, Nonconvex optimization and Its Applications, Vol. 88, Springer-Verlag, Berlin.
11. De la Cal, J., and Carcamo J. 2006. Multidimensional Hermite-Hadamard inequalities and the convex order. Journal of Mathematical Analysis and Applications, 324: 248-261.
12. Nikodem, K., On convex stochastic processes, Aequat. Math., 1980, 20, 184-197
13. Shaked, M., Shanthikumar, J.G., Stochastic convexity and its applications, Advances in Applied Probability, 1988, 20, 427-446
14. Skowronski, A., On some properties of J-convex stochastic processes, Aequat. Math., 1992, 44, 249-258.
15. Kotrys, D., Hermite-Hadamard inequality for convex stochastic processes, Aequat. Math., 2012, 83, 143-151.
16. Set, E., Sarıkaya, M.Z., Tomar, M., Hermite-Hadamard inequalities for coordinates convex stochastic processes, Mathematica Aeterna, 2015, 5 (1.2), 363-382.
17. V. Karahan and N. Okur, "Hermite-Hadamard Type Inequalities for Convex Stochastic Processes On n-Coordinates", Turk. J. Math. Comput. Sci., 10, pp.256-262, 2018.
18. N. Okur, and V. Karahan "Hermite-Hadamard Type Inequalities for s-Convex Stochastic Processes On n-Coordinates", Communications Faculty of Sciences University of Ankara Series A1-Mathematics and Statistics., submitted, 2019.
19. V. Karahan and N. Okur, "New Integral Inequalities of Hadamard's for Harmonically Convex Stochastic Processes on n-coordinates", ICAAMM19 Abstract Book., pp.202, 2019.
20. V. Karahan and N. Okur, "Some Generalized Hermite-Hadamard Type Integral Inequalities for φ-Convex Stochastic Processes on n-coordinates", ICAAMM19 Abstract Book., pp.201, 2019.
21. Akdemir G.H., Okur Bekar, N., & İşcan, İ., On Preinvexity for Stochastic Processes, Statistics, Journal of the Turkish Statistical Association, 7(1), 15-22, 2014.
22. N. Okur B., H.G. Akdemir, I. Iscan, "Some extensions of preinvexity for stochastic processes", Comp. Analysis, Springer Proceedings in Mathematics & Statistics, Springer, New York, 155, pp.259-270, 2016.
23. N. Okur, "Generalized Hadamard's Inequalities for Two-Dimensional Preinvex Stochastic Processes", ICAAMM19 Abstract Book., pp.218, 2019.